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SOLUTION SKETCHES TO HOMEWORK 10

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Homework 10.1 (Riemann sphere)**.** The goal of this exercise is to introduce calculus "at infinity". We set $\widehat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, where for the moment ∞ is an abstract element. We say a sequence $(z_n)_{n\in\mathbf{N}}$ in $\widehat{\mathbf{C}}$ converges to a point $z \in \widehat{\mathbf{C}}$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ we have

- $z_n = \infty$ or $z_n \in \mathbf{C}$ yet $|z_n| \geq 1/\varepsilon$ provided $z = \infty$ and
- $z_n \in \mathbf{C}$ and $|z_n - z| \leq \varepsilon$ provided $z \in \mathbf{C}$.

Moreover, let $\mathbf{S}^2 := \{x \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the usual two-dimensional unit sphere and define the stereographic projection $P\colon \mathbf{S}^2 \to \widehat{\mathbf{C}}$ through

$$P(x) := \begin{cases} \dfrac{x_1}{1-x_3} + \mathrm{i}\dfrac{x_2}{1-x_3} & \text{if } x_3 \neq 1,\\ \infty & \text{otherwise.}\end{cases}$$

a. Show P is a homeomorphism (where continuity is tacitly understood as sequential continuity).
b. Conclude $\widehat{\mathbf{C}}$ is sequentially compact.

Solution. a. We first show injectivity. Assume $P(x) = P(y)$ for two given points $x, y \in \mathbf{S}^2$, then either $P(x) = \infty$ so that $x = y = (0,0,1)$ or

$$\frac{x_1}{1-x_3} = \frac{y_1}{1-y_3},$$
$$\frac{x_2}{1-x_3} = \frac{y_2}{1-y_3}.$$

Squaring both equalities, their sum yields

$$\frac{1-x_3^2}{(1-x_3)^2} = \frac{x_1^2+x_2^2}{(1-x_3)^2} = \frac{y_1^2+y_2^2}{(1-y_3)^2} = \frac{1-y_3^2}{(1-y_3^2)},$$

where we used $x_1^2 + x_2^2 + x_3^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 = 1$. We can further simplify the terms on both sides to deduce

$$\frac{1+x_3}{1-x_3} = \frac{1+y_3}{1-y_3}.$$

Note the assignment $t \mapsto (1+t)(1-t)^{-1}$ is strictly increasing on $(0,1)$. Hence, the above equality yields $x_3 = y_3$ and we conclude $x = y$.

We turn to surjectivity. It clearly suffices to find, given any $z \in \mathbf{C}$, a point $x \in \mathbf{S}^2$ with $P(x) = z$. Since $|P(x)|^2 = (1+x_3)(1-x_3)^{-1}$ we set $x_3 = (|z|^2-1)(1+|z|^2)^{-1}$, so that $|x_3| < 1$. In conclusion we let

$$x_1 = \Re z\Big[1 - \frac{|z|^2-1}{1+|z|^2}\Big] = \Re z\frac{2}{1+|z|^2},$$
$$x_2 = \Im z\Big[1 - \frac{|z|^2-1}{1+|z|^2}\Big] = \Im z\frac{2}{1+|z|^2},$$

By a direct calculation, we find $x_1^2 + x_2^2 + x_3^2 = 1$ and $P(x) = z$.

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In particular, the inverse of P is given by

$$P^{-1}(z) = \begin{cases} \frac{1}{1+|z|^2}(2\Re z, 2\Im z, |z|^2 - 1) & \text{if } z \in \mathbf{C}, \\ (0,0,1) & \text{if } z = \infty. \end{cases}$$

We next show P and P^{-1} are sequentially continuous. Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in $\mathbb{S}^2$ converging to some $x \in \mathbf{S}^2$. If $x \neq (0,0,1)$ then clearly $P(x_n) \to P(x)$ as $n \to \infty$. If $x = (0,0,1)$, we may and will assume without loss of generality that $x_n \neq x$ for every $n \in \mathbb{N}$. Then $|P(x_n)| \to \infty$ as $n \to \infty$, which shows $P(x_n) \to \infty$ as $n \to \infty$. Thus P is continuous. Conversely, let us assume $(z_n)_{n\in\mathbf{N}}$ is a sequence in $\widehat{\mathbf{C}}$ converging to a point $z \in \widehat{\mathbf{C}}$. If $z \in \mathbf{C}$ then again $P^{-1}(z_n) \to P^{-1}(z)$ as $n \to \infty$. If $z = \infty$ we may and will again assume $z_n \neq \infty$ for every $n \in \mathbf{N}$. Then by definition $|z_n| \to \infty$ as $n \to \infty$, which entails $P^{-1}(z_n) = (0,0,1)$ thanks to the inequality $|\Re z|, |\Im z| \leq |z|$ for every $z \in \mathbf{C}$.

b. In order to show sequential compactness, it suffices to note $\widehat{\mathbf{C}}$ is the image of the sequentially compact set $\mathbf{S}^2$ under the continuous function P.

Homework 10.2 (Open mapping theorem for the Riemann sphere)**.** Let $\widehat{D} \subset \widehat{\mathbf{C}}$ be a domain and let $f\colon \widehat{D} \to \widehat{\mathbf{C}}$ be holomorphic and nonconstant. Show $f(\widehat{D})$ is again a domain.

Solution. Since f is continuous it follows $f(\widehat{D})$ is again a path-connected set. We claim it is an open set. Given $w \in f(\widehat{D})$, to find a neighborhood we distinguish several cases.

If $w \in \mathbf{C}$ and there exists $z \in \widehat{D} \setminus \{\infty\}$ with $f(z) = w$ then by continuity there exists $r > 0$ such that $B_r(z) \subset \widehat{D}$ and $f(B_r(z)) \in \mathbf{C}$. By the identity theorem it follows that f is nonconstant on $B_r(z)$. Hence by the standard open mapping theorem there exists $\varepsilon > 0$ such that $B_\varepsilon(w) \subset f(B_r(z)) \subset f(\widehat{D})$.

If $w = \infty$ and there exists $z \in \widehat{D} \setminus \{\infty\}$ with $f(z) = w$ we repeat the argument with the nonconstant holomorphic function $1/f$ on an appropriate ball $B_r(z)$. This yields there is $\varepsilon > 0$ such that for every $y \in B_\varepsilon(0)$ there exists $z' \in B_r(z)$ such that $1/f(z') = y$. (Recall that $1/\infty = 0$.) Rearranging terms this yields $\widehat{\mathbf{C}} \setminus \bar{B}_\varepsilon(0) \subset f(B_r(z))$. The left-hand side set is an open neighborhood of ∞.

If $w \in \mathbf{C}$ and $f(\infty) = w$ we consider the holomorphic nonconstant assignment $z \mapsto f(1/z)$ on $B_r(0)$, which is well-defined for $r > 0$ small enough, since $\widehat{D}$ is open. Again we deduce there exists $\varepsilon > 0$ such that $B_\varepsilon(w) \subset f(\widehat{D})$.

In the remaining case $w = \infty = f(\infty)$, we argue analogously by considering the assignment $z \mapsto 1/f(1/z)$.

In all four cases we conclude there exists a neighborhood $N \subset \widehat{\mathbf{C}}$ of w with the property that $N \subset f(\widehat{D})$. This concludes the proof.

Homework 10.3 (Extension of entire functions*)**.** In this exercise we show polynomials are the only entire functions that can be extended to the Riemann sphere in a holomorphic way.

a. Let $P\colon \mathbf{C} \to \mathbf{C}$ be a nonconstant polynomial. Show that setting $P(\infty) := \infty$ defines a holomorphic extension $P\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$.
b. Show if $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is holomorphic and satisfies $f(\mathbf{C}) \subset \mathbf{C}$ then f is a polynomial[1].

Homework 10.4 (Holomorphic functions on the Riemann sphere are rational)**.** Let the function $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be holomorphic. Show f is a rational function[2], i.e. there are polynomials $P, Q\colon \mathbf{C} \to \mathbf{C}$ such that for every $z \in \mathbf{C} \setminus \{f = \infty\}$,

$$f(z) = \frac{P(z)}{Q(z)}.$$

[1]**Hint.** Consider the assignment $z \mapsto f(1/z)$ and its singularity at 0.

[2]**Hint.** You may need the following generalized Liouville theorem. If $g\colon \mathbf{C} \to \mathbf{C}$ is holomorphic and there exist $R > 0$ and $n \in \mathbf{N}$ such that $|g(z)| \leq R|z|^n$ for every $z \in \bar{B}_R(0)$, then g is a polynomial no larger than n.

Solution. If f is constant, there is nothing to prove. Moreover, by Homework 10.3 we may and will assume without loss of generality that $f(z) = \infty$ for some $z \in \mathbf{C}$. Since $\widehat{\mathbf{C}}$ is compact, the identity theorem on $\widehat{\mathbf{C}}$ implies $f^{-1}(\infty) \cap \mathbf{C}$ is a finite set $\{a_1, \dots, a_n\}$, where $n \in \mathbf{N}$. At each point a_i the function $1/f$ has a zero of order $k_i \in \mathbf{N}$. This implies that the function f has pole of order k_i in a_i. Define the polynomial

$$Q(z) := \prod_{i=1}^{n} (z - a_i)^{k_i}.$$

Then the product fQ, defined on $\mathbf{C} \setminus \{a_1, \dots, a_n\}$, is holomorphic and each singularity is removable. Hence it can be extended to an entire function. If $f(\infty) \in \mathbf{C}$, there exists $R > 0$ such that $|f(z)| \leq R$ for every $z \in \mathbf{C}$ with $|z| \geq R$. The generalized Liouville theorem implies fQ is a polynomial. If $f(\infty) = \infty$ it follows that $fQ(\infty) := \infty$ defines a holomorphic extension. Indeed, this is just a reformulation of the product rule since by Homework 10.3 the polynomial Q is holomorphic at ∞ with value ∞. From Homework 10.3 we deduce again that fQ is a polynomial[3].

[3]**Remark.** We distinguished the two cases $f(\infty) = \infty$ and $f(\infty) \in \mathbf{C}$ because $f(\infty) = 0$ possibly requires a different extension. More generally, the product of two functions with values in $\widehat{\mathbf{C}}$ is not well-defined in general.